Upper bounds for Z_1-eigenvalues of generalized Hilbert tensors *

Juan Meng[†], Yisheng Song[‡]

Abstract

In this paper, we introduce the concept of Z_1-eigenvalue to infinite dimensional generalized Hilbert tensors (hypermatrix) $\mathcal{H}_\lambda^\infty = (\mathcal{H}_{i_1 i_2 \cdots i_m})$,

$$\mathcal{H}_{i_1 i_2 \cdots i_m} = \frac{1}{i_1 + i_2 + \cdots i_m + \lambda},\ \lambda \in \mathbb{R} \setminus \mathbb{Z}^-;\ i_1, i_2, \cdots, i_m = 0, 1, 2, \cdots, n, \cdots,$$

and proved that its Z_1-spectral radius is not larger than π for $\lambda > \frac{1}{2}$, and is at most $\frac{\pi}{\sin \lambda\pi}$ for $\frac{1}{2} \geq \lambda > 0$. Besides, the upper bound of Z_1-spectral radius of an mth-order n-dimensional generalized Hilbert tensor $\mathcal{H}_\lambda^n$ is obtained also, and such a bound only depends on n and λ.

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1 Introduction

A generalized Hilbert matrix has the form [13]:

$$H_\lambda^\infty = \left(\frac{1}{i + j + \lambda}\right)_{i,j \in \mathbb{Z}^+} \tag{1.1}$$

where $\mathbb{Z}^+$ ($\mathbb{Z}^-$) is the set of all non-negative (non-positive) integers and $\lambda \in \mathbb{R} \setminus \mathbb{Z}^-$. Denote such a Hilbert matrix with $i, j \in I_n = \{0, 1, 2, \cdots, n\}$ by H_λ^n. When $\lambda = 1$, such a matrix is called Hilbert matrix, which was introduced by Hilbert [12]. Choi [6] and Ingham [14] proved that Hilbert matrix H_1^∞ is a bounded linear operator (but not compact operator)

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†School of Mathematics and Information Science, Henan Normal University, XinXiang HeNan, P.R. China, 453007. Email: 1015791785@qq.com

‡Corresponding author. School of Mathematics and Information Science and Henan Engineering Laboratory for Big Data Statistical Analysis and Optimal Control, Henan Normal University, XinXiang HeNan, P.R. China, 453007. Email: songyisheng@htu.cn.

from Hilbert space l^2 into itself. Magnus [18] and Kato [15] studied the spectral properties of H_1^∞. Frazer [7] and Taussky [29] discussed some nice properties of n-dimensional Hilbert matrix H_1^n. Rosenblum [23] showed that for a real $\lambda < 1$, H_λ^∞ defines a bounded operator on l^p for $2 < p < \infty$ and that $\pi \sec \pi u$ is an eigenvalue of H_λ^∞ for $|\Re u| < \frac{1}{2} - \frac{1}{p}$. For each non-integer complex number λ, Aleman, Montes-Rodríguez, Sarafoleanu [1] showed that H_λ^∞ defines a bounded linear operator on the Hardy spaces H^p $(1 < p < \infty)$.

As a natural extension of a generalized Hilbert matrix, the generalized Hilbert tensor (hypermatrix) was introduced by Mei and Song [24]. For each $\lambda \in \mathbb{R} \setminus \mathbb{Z}^-$, the entries of an mth-order infinite dimensional generalized Hilbert tensor $\mathcal{H}_\lambda^\infty = (\mathcal{H}_{i_1 i_2 \cdots i_m})$ are defined by

$$\mathcal{H}_{i_1 i_2 \cdots i_m} = \frac{1}{i_1 + i_2 + \cdots i_m + \lambda}, \; i_1, i_2, \cdots, i_m = 0, 1, 2, \cdots, n, \cdots . \tag{1.2}$$

They showed $\mathcal{H}_\lambda^\infty$ defines a bounded and positively $(m-1)$-homogeneous operator from l^1 into l^p $(1 < p < \infty)$. Song and Qi [25] studied the operator properties of Hilbert tensors $\mathcal{H}_1^\infty$ and the spectral properties of $\mathcal{H}_1^n$. Such a tensor, $\mathcal{H}_\lambda^\infty$ may be refered to as a Hankel tensor with $v = (1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots)$. The concept of Hankel tensor was introduced by Qi [22]. For more further research of Hankel tensors, see Qi [22], Chen and Qi [5], Xu [31]. Denote such an mth-order n-dimensional generalized Hilbert tensor by $\mathcal{H}_\lambda^n$.

For a real vector $x = (x_1, x_2, \cdots, x_n, x_{n+1}, \cdots) \in l^1$, $\mathcal{H}_\lambda^\infty x^{m-1}$ is an infinite dimensional vector with its ith component defined by

$$(\mathcal{H}_\lambda^\infty x^{m-1})_i = \sum_{i_2, \cdots, i_m = 0}^{\infty} \frac{x_{i_2} \cdots x_{i_m}}{i + i_2 + \cdots + i_m + \lambda}, \lambda \in \mathbb{R} \setminus \mathbb{Z}^-; \; i = 0, 1, 2, \cdots . \tag{1.3}$$

Accordingly, $\mathcal{H}_\lambda^\infty x^m$ is given by

$$\mathcal{H}_\lambda^\infty x^m = \sum_{i_1, i_2, \cdots, i_m = 0}^{\infty} \frac{x_{i_1} x_{i_2} \cdots x_{i_m}}{i_1 + i_2 + \cdots + i_m + \lambda}, \lambda \in \mathbb{R} \setminus \mathbb{Z}^-. \tag{1.4}$$

Mei and Song [24] proved that $\mathcal{H}_\lambda^\infty x^m < \infty$ and $\mathcal{H}_\lambda^\infty x^{m-1} \in l^p$ $(1 < p < \infty)$ for all real vector $x \in l^1$.

In this paper, we will introduce the concept of Z_1-eigenvalue μ for an mth-order infinite dimensional generalized Hilbert tensor $\mathcal{H}_\lambda^\infty$ and will study some upper bounds of Z_1-spectral radius for infinite dimensional generalized Hilbert tensor $\mathcal{H}_\lambda^\infty$ and n-dimensional generalized Hilbert tensor $\mathcal{H}_\lambda^n$.

In Section 2, we will give some Lemmas and basic conclusions, and introduce the concept of Z_1-eigenvalue. In Section 3, with the help of the Hilbert type inequalities, the upper bound of Z_1–spectral radius of $\mathcal{H}_\lambda^\infty$ with $\lambda > 0$ is at most π when $\lambda > \frac{1}{2}$, and is not larger than $\frac{\pi}{\sin \lambda\pi}$ when $0 < \lambda \leq \frac{1}{2}$. Furthermore, for each Z_1-eigenvalue μ of $\mathcal{H}_\lambda^n$, $|\mu|$ is smaller than or equal to $C(n, \lambda)$, where $C(n, \lambda)$ only depends on the structured coefficient λ of generalized Hilbert tensor and the dimensionality n of European space.

2 Preliminaries and Basic Results

For $0 < p < \infty$, l^p is a space consisting of all real number sequences $x = (x_i)_{i=1}^{+\infty}$ satisfying $\sum\limits_{i=1}^{+\infty} |x_i|^p < \infty$. If $p \geq 1$, then a norm on l^p is defined by

$$\|x\|_{l^p} = \left(\sum_{i=1}^{+\infty} |x_i|^p \right)^{\frac{1}{p}}.$$

It is well known that l^2 is a Hilbert space with the inner product

$$\langle x, y \rangle = \sum_{i=0}^{+\infty} x_i y_i.$$

Clearly, $\|x\|_{l^2} = \sqrt{\langle x, x \rangle}$.

For $p \geq 1$, a norm $\mathbb{R}^n$ can be defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}.$$

It is well known that

$$\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2. \tag{2.1}$$

The following Hilbert type inequalities were proved by Frazer [7] on $\mathbb{R}^n$ and Ingham [14] on l^2, respectively.

Lemma 2.1. *(Frazer [7]) Let $x = (x_1, x_2, \cdots, x_n)^\top \in \mathbb{R}^n$. Then*

$$\sum_{i=0}^{n} \sum_{j=0}^{n} \frac{|x_i||x_j|}{i+j+1} \leq (n \sin \frac{\pi}{n}) \sum_{k=0}^{n} x_k^2 = \|x\|_2^2 n \sin \frac{\pi}{n}, \tag{2.2}$$

Lemma 2.2. *(Ingham [14]) Let $x = (x_1, x_2, \cdots, x_n, \cdots)^\top \in l^2$ and $a > 0$. Then*

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{|x_i||x_j|}{i+j+a} \leq M(a) \sum_{k=0}^{\infty} x_k^2 = M(a)\|x\|_{l^2}^2, \tag{2.3}$$

where

$$M(a) = \begin{cases} \frac{\pi}{\sin a\pi}, & 0 < a \leq \frac{1}{2}; \\ \pi, & a > \frac{1}{2}. \end{cases}$$

An m-order n-dimensional tensor (hypermatrix) $\mathcal{A} = (a_{i_1 \cdots i_m})$ is a multi-array of real entries $a_{i_1 \cdots i_m} \in \mathbb{R}$, where $i_j \in I_n = \{1, 2, \cdots, n\}$ for $j \in [m] = \{1, 2, \cdots, m\}$. We use $T_{m,n}$ denote the set of all real mth-order n-dimensional tensors. Then $\mathcal{A} \in T_{m,n}$ is called a symmetric tensor if the entries $a_{i_1 \cdots i_m}$are invariant under any permutation of their indices. $\mathcal{A} \in T_{m,n}$ is called nonnegative (positive) if $a_{i_1 i_2 \cdots i_m} \geq 0 (a_{i_1 i_2 \cdots i_m} > 0)$ for all $i_1, i_2, \cdots, i_m$.

Definition 2.1. (Chang and Zhang [2]) Let $\mathcal{A} \in T_{m,n}$. A number $\mu \in \mathbb{R}$ is called Z_1-eigenvalue of $\mathcal{A}$ if there is a real vector x such that

$$\begin{cases} \mathcal{A}x^{m-1} = \mu x \\ \|x\|_1 = 1 \end{cases} \tag{2.4}$$

and call such a vector x an Z_1-eigenvector associated with μ.

For the concepts of eigenvalues of higher order tensors, Qi [19, 20] first used and introduced them for symmetric tensors, and Lim [17] independently introduced this notion but restricted x to be a real vector and λ to be a real number. Subsequently, the spectral properties of nonnegative matrices had been generalized to n-dimensional nonnegative tensors under various conditions by Chang et al. [3, 4], He and Huang [9], He [10], He et al. [11], Li et al. [16], Qi [21], Song and Qi [26, 27], Wang et al. [30], Yang and Yang [32, 33] and references therein. The notion of Z_1-eigenvalue was introduced by Chang and Zhang [2] for higher Markov chains. Now we introduce it to infinite dimensional generalized Hilbert tensors.

Let

$$T_\infty x = \begin{cases} \|x\|_{l^1}^{2-m}\mathcal{H}_\lambda^\infty x^{m-1}, & x \neq \theta \\ \theta, & x = \theta, \end{cases} \tag{2.5}$$

where $\theta = (0, 0, \cdots, 0, \cdots)$. Mei and Song [24] first used the concept of the operator T_∞ induced by a generalized Hilbert tensor $\mathcal{H}_\lambda^\infty$ and showed T_∞ is a bounded and positively homogeneous operator from l^1 into l^p $(1 < p < \infty)$. Then T_∞ is refered to as a bounded and positively homogeneous operator from l^2 into l^2. So, the concept of Z_1-eigenvalue may be introduced to the infinite dimensional Hilbert tensor $\mathcal{H}_\lambda^\infty$.

Definition 2.2. Let $\mathcal{H}_\lambda^\infty$ be an mth-order infinite dimensional generalized Hilbert tensor. A real number μ is called a Z_1-eigenvalue of $\mathcal{H}_\lambda^\infty$ if there exists a nonzero vector $x \in l^2$ satisfying

$$T_\infty x = \|x\|_{l^1}^{2-m}\mathcal{H}_\lambda^\infty x^{m-1} = \mu x. \tag{2.6}$$

Such a vector x is called an Z_1-eigenvector associated with μ.

3 Main Results

Theorem 3.1. *Let $\mathcal{H}_\lambda^n$ be an mth-order n-dimensional generalized Hilbert tensor. Then*

$$|\mu| \le C(n, \lambda) \text{ for all } Z_1\text{-eigenvalue } \mu \text{ of } \mathcal{H}_\lambda^n,$$

where $[\lambda]$ is the largest integer not exceeding λ and

$$C(n,\lambda) = \begin{cases} n \sin\frac{\pi}{n}, & \lambda \ge 1; \\ \frac{n}{\lambda}, & 1 > \lambda > 0; \\ \frac{n}{\min\{\lambda-[\lambda], 1+[\lambda]-\lambda\}}, & -mn < \lambda < 0; \\ \frac{n}{-mn-a}, & \lambda < -mn. \end{cases}$$

Proof. For $\lambda \geq 1$, it follows from Lemma 2.1 that for all nonzero vector $x \in \mathbb{R}^n$,

$$\begin{aligned}
|\mathcal{H}_\lambda^n x^m| &= \left| \sum_{i_1,i_2,\cdots,i_m=0}^{n} \frac{x_{i_1}x_{i_2}\cdots x_{i_m}}{i_1+i_2+\cdots+i_m+\lambda} \right| \\
&\leq \sum_{i_1,\cdots,i_m=0}^{n} \frac{|x_{i_1}x_{i_2}\cdots x_{i_m}|}{i_1+i_2+\underbrace{0+\cdots+0}_{m-2}+\lambda} \\
&= \sum_{i_1,i_2,\cdots,i_m=0}^{n} \frac{|x_{i_1}||x_{i_2}|\cdots|x_{i_m}|}{i_1+i_2+\lambda} \\
&= \left(\sum_{i_1=0}^{n}\sum_{i_2=0}^{n} \frac{|x_{i_1}||x_{i_2}|}{i_1+i_2+\lambda} \right) \sum_{i_3,i_4,\cdots,i_m=0}^{n} |x_{i_3}||x_{i_4}|\cdots|x_{i_m}| \\
&\leq \left(\sum_{i_1=0}^{n}\sum_{i_2=0}^{n} \frac{|x_{i_1}||x_{i_2}|}{i_1+i_2+1} \right) \sum_{i_3,i_4,\cdots,i_m=0}^{n} |x_{i_3}||x_{i_4}|\cdots|x_{i_m}| \\
&\leq (\|x\|_2^2 n \sin\frac{\pi}{n}) \left(\sum_{i=0}^{n} |x_i| \right)^{m-2} \\
&= \|x\|_2^2 \|x\|_1^{m-2} n \sin\frac{\pi}{n}.
\end{aligned}$$

That is,

$$|\mathcal{H}_\lambda^n x^m| \leq \|x\|_2^2 \|x\|_1^{m-2} n \sin\frac{\pi}{n}. \tag{3.1}$$

Since μ is a Z_1-eigenvalue of $\mathcal{H}_\lambda^n$, then there exists a nonzero vector x such that

$$\mathcal{H}_\lambda^n x^{m-1} = \mu x \text{ and } \|x\|_1 = 1. \tag{3.2}$$

Thus, we have,

$$|\mu x^\top x| = |x^\top (\mathcal{H}_\lambda^n x^{m-1})| = |\mathcal{H}_\lambda^n x^m| \leq \|x\|_2^2 \|x\|_1^{m-2} n \sin\frac{\pi}{n},$$

and then,

$$|\mu| \|x\|_2^2 \leq \|x\|_2^2 \|x\|_1^{m-2} n \sin\frac{\pi}{n}.$$

As a result,

$$|\mu| \leq n \sin\frac{\pi}{n}. \tag{3.3}$$

For all $\lambda \in \mathbb{R} \setminus \mathbb{Z}^-$ with $\lambda < 1$, it is obvious that for $1 > \lambda > 0$,

$$\min_{i_1,\cdots,i_m \in I_n} |i_1+i_2+\cdots+i_m+\lambda| = \lambda.$$

For $-mn < \lambda < 0$, there exist some positive integers $i_1', i_2', \cdots, i_m'$ and $i_1'', i_2'', \cdots, i_m''$ such that

$$i_1'+i_2'+\cdots+i_m' = -[\lambda] \text{ and } i_1''+i_2''+\cdots+i_m'' = -[\lambda]-1,$$

and hence,

$$\min_{i_1,\cdots,i_m\in I_n} |i_1+i_2+\cdots+i_m+\lambda| = \min\{\lambda-[\lambda], \lambda-(-[\lambda]-1)\}.$$

For $\lambda < -mn$, we also have,

$$\min_{i_1,\cdots,i_m\in I_n} |i_1+i_2+\cdots+i_m+\lambda| = |mn+\lambda| = -mn-\lambda.$$

Therefore, we have for $\lambda \in \mathbb{R}\setminus\mathbb{Z}^-$ with $\lambda < 1$,

$$\frac{1}{|i_1+i_2+\cdots+i_m+\lambda|} \leq N(\lambda) = \begin{cases} \frac{1}{\lambda}, & 1>\lambda>0; \\ \frac{1}{\min\{\lambda-[\lambda],1+[\lambda]-\lambda\}}, & -mn<\lambda<0; \\ \frac{1}{-mn-a}, & \lambda<-mn. \end{cases}$$

Then, for all nonzero vector $x \in \mathbb{R}^n$, we have

$$\begin{aligned} |\mathcal{H}_\lambda^n x^m| &= \left| \sum_{i_1,i_2,\cdots,i_m=0}^{n} \frac{x_{i_1}x_{i_2}\cdots x_{i_m}}{i_1+i_2+\cdots+i_m+\lambda} \right| \\ &\leq \sum_{i_1,\cdots,i_m=0}^{n} \frac{|x_{i_1}x_{i_2}\cdots x_{i_m}|}{|i_1+i_2+\cdots+i_m+\lambda|} \\ &\leq N(\lambda) \sum_{i_1,i_2,\cdots,i_m=0}^{n} |x_{i_1}||x_{i_2}|\cdots|x_{i_m}| \\ &= N(\lambda)\left(\sum_{i=0}^{n}|x_i|\right)^m = N(\lambda)\|x\|_1^m. \end{aligned}$$

For each Z_1-eigenvalue μ of $\mathcal{H}_\lambda^n$ with its eigenvector x, from (3.2) and $\|x\|_1 \leq \sqrt{n}\|x\|_2$, it follows taht

$$|\mu|(\frac{1}{n}\|x\|_1^2) \leq |\mu|\|x\|_2^2 = |\mathcal{H}_\lambda^n x^m| \leq N(\lambda)\|x\|_1^m,$$

and hence,

$$|\mu| \leq nN(\lambda).$$

This completes the proof. □

When $\lambda = 1$, the following conclusion of Hilbert tensor is easily obtained. Also see Song and Qi [25] for the conclusions about H-eigenvalue and Z-eigenvalue of such a tensor.

Corollary 3.2. *Let $\mathcal{H}$ be an mth-order n-dimensional Hilbert tensor. Then for all Z_1-eigenvalue μ of $\mathcal{H}$,*

$$|\mu| \leq n\sin\frac{\pi}{n}.$$

Theorem 3.3. *Let $\mathcal{H}_\lambda^\infty$ be an mth-order infinite dimensional generalized Hilbert tensor. Assume $\lambda > 0$, then for Z_1-eigenvalue μ of $\mathcal{H}_\lambda^\infty$,*

$$|\mu| \leq M(\lambda) = \begin{cases} \frac{\pi}{\sin\lambda\pi}, & 0<\lambda\leq\frac{1}{2}; \\ \pi, & \lambda>\frac{1}{2}. \end{cases}$$

Proof. For $x \in l^2$, it follows from Lemma 2.2 that

$$\begin{aligned}
|\langle x, \mathcal{H}_\lambda^\infty x^{m-1}\rangle| = |\mathcal{H}_\lambda^\infty x^m| &= \left| \sum_{i_1,i_2,\cdots,i_m=0}^{+\infty} \frac{x_{i_1}x_{i_2}\cdots x_{i_m}}{i_1+i_2+\cdots+i_m+\lambda} \right| \\
&\le \sum_{i_1,\cdots,i_m=0}^{+\infty} \frac{|x_{i_1}x_{i_2}\cdots x_{i_m}|}{i_1+i_2+\underbrace{0+\cdots+0}_{m-2}+\lambda} \\
&= \sum_{i_1,i_2,\cdots,i_m=0}^{+\infty} \frac{|x_{i_1}||x_{i_2}|\cdots|x_{i_m}|}{i_1+i_2+\lambda} \\
&= \left(\sum_{i_1=0}^{+\infty}\sum_{i_2=0}^{+\infty} \frac{|x_{i_1}||x_{i_2}|}{i_1+i_2+\lambda} \right) \sum_{i_3,i_4,\cdots,i_m=0}^{+\infty} |x_{i_3}||x_{i_4}|\cdots|x_{i_m}| \\
&= \left(\sum_{i_1=0}^{+\infty}\sum_{i_2=0}^{\infty} \frac{|x_{i_1}||x_{i_2}|}{i_1+i_2+\lambda} \right) \left(\sum_{i=0}^{+\infty} |x_i| \right)^{m-2} \\
&\le M(\lambda)\|x\|_{l^2}^2 \|x\|_{l^1}^{m-2},
\end{aligned}$$

and so,

$$|\langle x, T_\infty x\rangle| = |\langle x, \|x\|_1^{2-m}\mathcal{H}_\lambda^\infty x^{m-1}\rangle| = \|x\|_{l^1}^{2-m}|\mathcal{H}_\lambda^\infty x^m| \le M(\lambda)\|x\|_{l^2}^2. \tag{3.4}$$

For each Z_1-eigenvalue μ of $\mathcal{H}_\lambda^\infty$, there exists a nonzero vector $x \in l^2$ such that

$$T_\infty x = \|x\|_{l^1}^{2-m}\mathcal{H}_\lambda^\infty x^{m-1} = \mu x,$$

and so,

$$\mu\|x\|_{l^2}^2 = \mu\langle x, x\rangle = \langle x, \|x\|_{l^1}^{2-m}\mathcal{H}_\lambda^\infty x^{m-1}\rangle = \|x\|_{l^1}^{2-m}\mathcal{H}_\lambda^\infty x^m.$$

Therefore, we have

$$|\mu|\|x\|_{l^2}^2 = \|x\|_{l^1}^{2-m}|\mathcal{H}_\lambda^\infty x^m| \le M(\lambda)\|x\|_{l^2}^2,$$

and then,

$$|\mu| \le M(\lambda).$$

This completes the proof. □

When $\lambda = 1$, the following conclusion of infinite dimensional Hilbert tensor is easily obtained.

Corollary 3.4. *Let $\mathcal{H}_\infty$ be an mth-order infinite dimensional Hilbert tensor. Then for all Z_1-eigenvalue μ of $\mathcal{H}_\infty$,*

$$|\mu| \le \pi.$$

Remark 3.1. *(i) In Theorem 3.1, the upper bound of Z_1-eigenvalue of $\mathcal{H}_\lambda^n$ are showed. However the upper bound may not be the best. Then which number is its best upper bounds?*

(ii) In Theorem 3.3, the upper bound of Z_1-eigenvalue of $\mathcal{H}_\lambda^\infty$ are showed for $\lambda > 0$, then for $\lambda < 0$ with $\lambda \in \mathbb{R} \setminus \mathbb{Z}^-$, it is unknown whether have similar conclusions or not. And it is not clear whether the upper bound may be attained or cannot be attained.

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