The obstacle problem for non-coercive equations with lower order term and L^1-data

Jun Zheng

School of Mathematics, Southwest Jiaotong University, Chengdu 611756, China

Abstract

The aim of this paper is to study the obstacle problem associated with an elliptic operator having degenerate coercivity, and with a low order term and L^1-data. We prove the existence of an entropy solution to the obstacle problem and show its continuous dependence on the L^1-data in $W^{1,q}(\Omega)$ with some $q > 1$.

Key words: obstacle problem; non-coercive equation; entropy solution; L^1-data; lower order term.

1 Introduction

1.1 Problem setting and main result

Let Ω be a bounded domain in $\mathbb{R}^N(N \geq 2)$, $1 < p < +\infty$ and $\theta \geq 0$. Given functions $g,\ \psi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and data $f \in L^1(\Omega)$, the aim of this paper is to study the obstacle problem for nonlinear non-coercive elliptic equations with lower order term, governed by the operator

$$Au = -\text{div}\,\frac{a(x, \nabla u)}{(1 + |u|)^{\theta(p-1)}} + b|u|^{r-2}u, \tag{1}$$

where $b > 0$ is a constant, and $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function, satisfying the following conditions:

$$a(x, \xi) \cdot \xi \geq \alpha|\xi|^p, \tag{2}$$

$$|a(x, \xi)| \leq \beta(j(x) + |\xi|^{p-1}), \tag{3}$$

$$(a(x, \xi) - a(x, \eta))(\xi - \eta) > 0, \tag{4}$$

$$|a(x, \xi) - a(x, \zeta)| \leq \gamma \begin{cases} |\xi - \zeta|^{p-1}, & \text{if } 1 < p < 2 \\ (1 + |\xi| + |\zeta|)^{p-2}|\xi - \zeta|, & \text{if } p \geq 2 \end{cases}, \tag{5}$$

for almost every x in Ω, and for every ξ, η, ζ in $\mathbb{R}^N$ with $\xi \neq \eta$, where $\alpha, \beta, \gamma > 0$ are constants, and j is a nonnegative function in $L^{p'}(\Omega)$.

If f has a fine regularity, i.e., $f \in W^{-1,p'}(\Omega)$, the obstacle problem corresponding to (f, ψ, g) can be formulated in terms of the inequality

$$\int_\Omega \frac{a(x, \nabla u)}{(1 + |u|)^{\theta(p-1)}} \cdot \nabla(u - v)\mathrm{d}x + \int_\Omega b|u|^{r-2}u(u - v)\mathrm{d}x \leq \int_\Omega f(u - v)\mathrm{d}x,\ \ \forall v \in K_{g,\psi} \cap L^\infty(\Omega), \tag{6}$$

Email address: zhengjun@swjtu.edu.cn (Jun Zheng).

whenever $1 \le r < p$ and the convex subset

$$K_{g,\psi} = \{v \in W^{1,p}(\Omega);\ v - g\ \in W_0^{1,p}(\Omega),\ v \ge \psi,\ \text{a.e. in } \Omega\}$$

is nonempty. However, if $f \in L^1(\Omega)$, (6) is not well-defined, and following [1,3,6] etc., we are led to the more general definition of a solution to the obstacle problem, using the truncation function

$$T_s(t) = \max\{-s, \min\{s, t\}\},\quad s, t \in \mathbb{R}.$$

Definition 1 *An entropy solution of the obstacle problem associated corresponding to operator A and functions (f, ψ, g) with $f \in L^1(\Omega)$ is a measurable function u such that $u \ge \psi$ a.e. in Ω, $\frac{a(x,\nabla u)}{(1+|u|)^{\theta(p-1)}} \in (L^1(\Omega))^N$, $|u|^{r-1} \in L^1(\Omega)$, and, for every $s > 0$, $T_s(u) - T_s(g) \in W_0^{1,p}(\Omega)$ and*

$$\int_\Omega \frac{a(x, \nabla u)}{(1 + |u|)^{\theta(p-1)}} \cdot \nabla(T_s(u - v))dx + \int_\Omega b|u|^{r-2}uT_s(u - v)dx \le \int_\Omega fT_s(u - v)dx,\ \ \forall v \in K_{g,\psi} \cap L^\infty(\Omega). \tag{7}$$

Observe that no global integrability condition is required on u nor on its gradient in the definition. As pointed out in [3,9], if $T_s(u) \in W^{1,p}(\Omega)$ for all $s > 0$, then there exists a unique measurable vector field $U : \Omega \to \mathbb{R}^N$ such that $\nabla(T_s(u)) = \chi_{\{|u|<s\}}U$ a.e. in Ω, $s > 0$, which, in fact, coincides with the standard distributional gradient of ∇u whenever $u \in W^{1,1}(\Omega)$.

Before stating the main result, we make some basic assumptions throughout this paper, i.e., without special statements, we always assume that

$$2 - \frac{1}{N} < p < N,\ 1 \le r < p,\ 0 \le \theta < \min\left\{\frac{N}{N-1} - \frac{1}{p-1}, \frac{p-r}{p-1}\right\}, b > 0,$$

and $\psi, g \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $(\psi - g)^+ \in W_0^{1,p}(\Omega)$ such that $K_{g,\psi} \neq \emptyset$. The following theorem is the main result obtained in this paper.

Theorem 1 *Let $f \in L^1(\Omega)$. Then there exists at least one entropy solution u of the obstacle problem associated with (f, ψ, g). In addition, u depends continuously on f, i.e., if $f_n \to f$ in $L^1(\Omega)$ and u_n is a solution to the obstacle problem associated with (f_n, ψ, g), then*

$$u_n \to u \text{ in } W^{1,q}(\Omega), \forall q \in \begin{cases} \left(\frac{N(r-1)}{N+r-1}, \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)}\right), & \text{if } \frac{2N-1}{N-1} \le r < p, \\ \left(1, \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)}\right), & \text{if } 1 \le r < \min\{\frac{2N-1}{N-1}, p\}. \end{cases} \tag{8}$$

1.2 Some comments and remarks

Consider the Dirichlet boundary value problem having a form

$$\begin{cases} -\text{div}\, \frac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}} + bu = f, \text{ in } \Omega, \\ u = 0, \qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{9}$$

with $p > 1, \theta \in (0, 1], b \ge 0, f \in L^1(\Omega)$. The item $\frac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}}$ is not coercive if u is very large. Due to this, the classical methods used to prove the existence of a solution for elliptic equations, i.e., [15], cannot be applied even if $b = 0$ and the data f is regular. Moreover, $\frac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}}$ and u and f are only in $L^1(\Omega)$, not in $W^{-1,p'}(\Omega)$. All these characteristics prevent us employing the duality argument [18] or nonlinear monotone operator theory [19] directly.

To overcome these difficulties, cutting the noncoercivity term and using the technique of approximation, a pseudomonotone and coercive differential operator on $W_0^{1,p}(\Omega)$ can be applied to establish *a priori* estimates on approximating solutions. As a result, existence of solutions, or entropy solutions, can be obtained by taking limitation for $f \in L^m(\Omega), m \geq 1$ and $b > 0$ due to the almost everywhere convergence for the gradients of the approximating solutions, see e.g., [4,7,10–12,16] (see also [1,2,8,13,14,17] for $b = 0$).

Motivated by the study on the non-coercive elliptic equations (9), we consider in this paper the obstacle problem governed by (1) and functions (f, ψ, g) with $f \in L^1(\Omega)$. We prove the existence of an entropy solution and show its continuous dependence on the L^1-data in $W^{1,q}(\Omega)$ with some $q \in (1, p)$.

In the following, we give some remarks on our main result and inequalities that will be needed in the proofs. Some notations are provided in the end of this subsection.

Remark 1 *(i)* $0 \leq \theta < \min\left\{\frac{N}{N-1} - \frac{1}{p-1}, \frac{p-r}{p-1}\right\} \Rightarrow r-1 < (1-\theta)(p-1) < \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)}$*. Therefore Theorem 1 guarantees* $|u|^{r-1} \in L^1(\Omega)$*, and the second integration in* (7) *makes sense.*

(ii) We will show that $\frac{a(x,\nabla u)}{(1+|u|)^{\theta(p-1)}} \in (L^1(\Omega))^N$ *in Proposition 4. Therefore, the first integration in* (7) *makes sense.*

(ii) $\left(\frac{N(r-1)}{N+r-1}, \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)}\right) \subset \left(1, \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)}\right)$ *if* $\frac{2N-1}{N-1} \leq r < p$*. Indeed,* $\theta < \frac{p-r}{p-1} + \frac{p(r-1)}{N(p-1)} \Leftrightarrow \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)} > \frac{N(r-1)}{N+r-1}$*, while* $\frac{2N-1}{N-1} \leq r$ *gives* $\frac{N(r-1)}{N+r-1} \geq 1$*. Thus* $u_n \to u$ *in* $W^{1,q}(\Omega)$ *for all* $q \in \left(1, \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)}\right)$*.*

(iii) $r-1 < \frac{Nq}{N-q}$*. Indeed, by* $1 \leq r < \frac{2N-1}{N-1}$*, there holds* $r-1 < \frac{N}{N-1} < \frac{Nq}{N-q}$ *for any* $q > 1$*, particularly, for* $q \in (1, \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)})$*. For* $r \geq \frac{2N-1}{N-1}$*, it suffices to note that* $q > \frac{N(r-1)}{N+r-1} \Leftrightarrow r-1 < \frac{Nq}{N-q}$*.*

(iv) $q < p$*. Indeed,* $0 \leq \theta < \frac{N}{N-1} - \frac{1}{p-1} < \frac{N-1}{p-1} \Rightarrow \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)} < p$*.*

Remark 2 *Checking proofs of this paper, one may find that, for* $b = 0$*,* (8) *holds with*

$$u_n \to u \text{ in } W^{1,q}(\Omega), \forall q \in \left(1, \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)}\right). \tag{10}$$

Indeed, it suffices to set $r = 1$ *in the proofs.*

Notations

$\|u\|_p = \|u\|_{L^p(\Omega)}$ is the norm of $L^p(\Omega)$, where $1 \leq p \leq \infty$. $\|u\|_{1,p} = \|u\|_{W^{1,p}(\Omega)}$ is the norm of $W^{1,p}(\Omega)$, where $1 < p < \infty$. $p' = \frac{p}{p-1}$ with $1 < p < \infty$. $u^+ = \max\{u, 0\}$, $u^- = (-u)^+$, if u is a real-valued function. C is a constant, which may be different from each other. $\{u > s\} = \{x \in \Omega; u(x) > s\}$. $\{u \leq s\} = \Omega \setminus \{u > s\}$. $\{u < s\} = \{x \in \Omega; u(x) < s\}$. $\{u \geq s\} = \Omega \setminus \{u < s\}$. $\{u = s\} = \{x \in \Omega; u(x) = s\}$. $\{t \leq u < s\} = \{u \geq t\} \cap \{u < s\}$. $\mathcal{L}^N$ is the Lebesgue measure of $\mathbb{R}^N$. $|E| = \mathcal{L}^N(E)$ for a measurable set E.

2 Lemmas on entropy solutions

It is worthy to note that, for any smooth function f_n, there exists at least one solution to the obstacle problem (6). Indeed, one can proceed exactly as in [1,12] to obtain $W^{1,p}-$solutions due to the assumptions (2)-(4) on a and $r - 1 < p$. These solutions, in particular, are also entropy solutions. In this section we establish several auxiliary results on convergence of sequences of entropy solutions when $f_n \to f$ in $L^1(\Omega)$.

Lemma 2 *Let* $v_0 \in K_{g,\psi} \cap L^\infty(\Omega)$*, and let* u *be an entropy solution of the obstacle problem associated with* (f, ψ, g)*. Then, we have*

$$\int_{\{|u|<t\}} \frac{|\nabla u|^p}{(1+|u|)^{\theta(p-1)}} dx \leq C(1+t^r), \ \forall t > 0,$$

where C *is a positive constant depending only on* $\alpha, \beta, p, r, b, \|j\|_{p'}, \|\nabla v_0\|_p, \|v_0\|_\infty$*, and* $\|f\|_1$*.*

Proof Take v_0 as a test function in (7). For t large enough such that $t - \|v_0\|_\infty > 0$, we get

$$\int_{\{|v_0-u|<t\}} \frac{a(x,\nabla u)\cdot\nabla u}{(1+|u|)^{\theta(p-1)}}\mathrm{d}x \le \int_{\{|v_0-u|<t\}} \frac{a(x,\nabla u)\cdot\nabla v_0}{(1+|u|)^{\theta(p-1)}}\mathrm{d}x + \int_\Omega (f-b|u|^{r-2}u)T_t(u-v_0)\mathrm{d}x. \tag{11}$$

We estimate each integration in the right-hand side of (11). It follows from (3) and Young's inequality with $\varepsilon > 0$ that

$$\begin{aligned}\int_{\{|v_0-u|<t\}} \frac{a(x,\nabla u)\cdot\nabla v_0}{(1+|u|)^{\theta(p-1)}}\mathrm{d}x \le& \int_{\{|v_0-u|<t\}} \frac{\beta(|j|+|\nabla u|^{p-1})\cdot|\nabla v_0|}{(1+|u|)^{\theta(p-1)}}\mathrm{d}x\\ \le& \int_{\{|v_0-u|<t\}} \frac{\beta\varepsilon(|j|^{p'}+|\nabla u|^p)}{(1+|u|)^{\theta(p-1)}}\mathrm{d}x + \int_{\{|v_0-u|<t\}} \frac{\beta C(\varepsilon)|\nabla v_0|^p}{(1+|u|)^{\theta(p-1)}}\mathrm{d}x\\ \le&\varepsilon \int_{\{|v_0-u|<t\}} \frac{|\nabla u|^p}{(1+|u|)^{\theta(p-1)}}\mathrm{d}x + C(\|j\|_{p'}^{p'} + \|\nabla v_0\|_p^p).\end{aligned} \tag{12}$$

$$-\int_\Omega b|u|^{r-2}uT_t(u-v_0)\mathrm{d}x = -\int_{\{|u-v_0|\le t\}} b|u|^{r-2}uT_t(u-v_0)\mathrm{d}x - \int_{\{|u-v_0|>t\}} b|u|^{r-2}uT_t(u-v_0)\mathrm{d}x. \tag{13}$$

Note that on the set $\{|u-v_0|\le t\}$,

$$||u|^{r-2}uT_t(u-v_0)| \le t|t+\|v_0\|_\infty|^{r-1} \le C(1+t^r), \tag{14}$$

where C is a constant depending only on r, $\|v_0\|_\infty$.
On the set $\{|u-v_0|>t\}$, we have $|u| \ge t - \|v_0\|_\infty > 0$, thus u and $T_t(u-v_0)$ have the same sign. It flows

$$-\int_{\{|u-v_0|>t\}} b|u|^{r-2}uT_t(u-v_0)\mathrm{d}x \le 0. \tag{15}$$

Combining (13)-(15), we get

$$-\int_\Omega b|u|^{r-2}uT_t(u-v_0)\mathrm{d}x \le C(1+t^r). \tag{16}$$

$$\begin{aligned}\int_{\{|v_0-u|<t\}} \frac{|\nabla u|^p}{(1+|u|)^{\theta(p-1)}}\mathrm{d}x \le& C(\|j\|_{p'}^{p'} + \|\nabla v_0\|_p^p + t\|f\|_1 + 1 + t^r)\\ \le& C(1+t^r).\end{aligned} \tag{17}$$

Replacing t with $t+\|v_0\|_\infty$ in (17) and noting that $\{|u|<t\} \subset \{|v_0-u|<t+\|v_0\|_\infty\}$, one may obtain the desired result.■

In the rest of this section, let $\{u_n\}$ be a sequence of entropy solutions of the obstacle problem associated with (f_n, ψ, g) and assume that

$$f_n \to f \ \text{ in } L^1(\Omega) \ \text{ and } \ \|f_n\|_1 \le \|f\|_1 + 1.$$

Lemma 3 *There exists a measurable function u such that $u_n \to u$ in measure, and $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W^{1,p}(\Omega)$ for any $k>0$. Thus $T_k(u_n) \to T_k(u)$ strongly in $L^p(\Omega)$ and a.e. in Ω.*

Proof Let s, t and ε be positive numbers. One may verify that for every $m, n \ge 1$,

$$\mathcal{L}^N(\{|u_n-u_m|>s\}) \le \mathcal{L}^N(\{|u_n|>t\}) + \mathcal{L}^N(\{|u_m|>t\}) + \mathcal{L}^N(\{|T_k(u_n)-T_k(u_m)|>s\}), \tag{18}$$

and

$$\mathcal{L}^N(\{|u_n|>t\})=\frac{1}{t^p}\int_{\{|u_n|>t\}}t^p\mathrm{d}x\leq\frac{1}{t^p}\int_\Omega|T_t(u_n)|^p\mathrm{d}x. \tag{19}$$

Due to $v_0=g+(\psi-g)^+\in K_{g,\psi}\cap L^\infty(\Omega)$, by Lemma 2, one has

$$\int_\Omega|\nabla T_t(u_n)|^p\mathrm{d}x=\int_{\{|u_n|<t\}}|\nabla u_n|^p\mathrm{d}x\leq C(1+t)^{\theta(p-1)}(1+t^r). \tag{20}$$

Note that $T_t(u_n)-T_t(g)\in W_0^{1,p}(\Omega)$. By (19), (20) and Poincaré's inequality, for every $t>\|g\|_\infty$ and for some positive constant C independent of n and t, there holds

$$\begin{aligned}\mathcal{L}^N(\{|u_n|>t\})\leq&\frac{1}{t^p}\int_\Omega|T_t(u_n)|^p\mathrm{d}x\\ \leq&\frac{2^{p-1}}{t^p}\int_\Omega|T_t(u_n)-T_t(g)|^p\mathrm{d}x+\frac{2^{p-1}}{t^p}\|g\|_p^p\\ \leq&\frac{C}{t^p}\int_\Omega|\nabla T_t(u_n)-\nabla T_t(g)|^p\mathrm{d}x+\frac{2^{p-1}}{t^p}\|g\|_p^p\\ \leq&\frac{C}{t^p}\int_\Omega|\nabla T_t(u_n)|^p\mathrm{d}x+\frac{C}{t^p}\|g\|_{1,p}^p\\ \leq&\frac{C(1+t^{r+\theta(p-1)})}{t^p}.\end{aligned}$$

Since $0\leq\theta<\frac{p-r}{p-1}$, there exists $t_\varepsilon>0$ such that

$$\mathcal{L}^N(\{|u_n|>t\})<\frac{\varepsilon}{3},\quad\forall\, t\geq t_\varepsilon,\ \forall\, n\geq 1. \tag{21}$$

Now we have as in (19)

$$\mathcal{L}^N(\{|T_{t_\varepsilon}(u_n)-T_{t_\varepsilon}(u_m)|>s\})=\frac{1}{s^p}\int_{\{|T_{t_\varepsilon}(u_n)-T_{t_\varepsilon}(u_m)|>s\}}s^p\mathrm{d}x\leq\frac{1}{s^p}\int_\Omega|T_{t_\varepsilon}(u_n)-T_{t_\varepsilon}(u_m)|^p\mathrm{d}x. \tag{22}$$

Using (20) and the fact that $T_t(u_n)-T_t(g)\in W_0^{1,p}(\Omega)$ again, we see that $\{T_{t_\varepsilon}(u_n)\}$ is a bounded sequence in $W^{1,p}(\Omega)$. Thus, up to a subsequence, $\{T_{t_\varepsilon}(u_n)\}$ converges strongly in $L^p(\Omega)$. Taking into account (22), there exists $n_0=n_0(t_\varepsilon,s)\geq 1$ such that

$$\mathcal{L}^N(\{|T_{t_\varepsilon}(u_n)-T_{t_\varepsilon}(u_m)|>s\})<\frac{\varepsilon}{3},\quad\forall\, n,m\geq n_0. \tag{23}$$

Combining (18), (21) and (23), we obtain

$$\mathcal{L}^N(\{|u_n-u_m|>s\})<\varepsilon,\quad\forall\, n,m\geq n_0.$$

Hence $\{u_n\}$ is a Cauchy sequence in measure, and therefore there exists a measurable function u such that $u_n\to u$ in measure. The remainder of the lemma is a consequence of the fact that $\{T_k(u_n)\}$ is a bounded sequence in $W^{1,p}(\Omega)$. ■

Proposition 4 *There exists a subsequence of $\{u_n\}$ and a measurable function u such that for each q given in* (8)*, we have*

$$u_n\to u\ \text{ strongly in } W^{1,q}(\Omega).$$

If moreover $0\leq\theta<\min\{\frac{1}{N-p+1},\frac{N}{N-1}-\frac{1}{p-1},\frac{p-r}{p-1}\}$, then

$$\frac{a(x,\nabla u_n)}{(1+|u_n|)^{\theta(p-1)}}\to\frac{a(x,\nabla u)}{(1+|u|)^{\theta(p-1)}}\ \text{ strongly in } (L^1(\Omega))^N.$$

To prove Proposition 4, we need two preliminary lemmas.

Lemma 5 *There exists a subsequence of $\{u_n\}$ and a measurable function u such that for each q given in* (8), *we have $u_n \rightharpoonup u$ weakly in $W^{1,q}(\Omega)$, and $u_n \to u$ strongly in $L^q(\Omega)$.*

Proof Let $k > 0$ and $n \geq 1$. Define $D_k = \{|u_n| \leq k\}$ and $B_k = \{k \leq |u_n| < k+1\}$. Using Lemma 2 with $v_0 = g+(\psi-g)^+$, we get

$$\int_{D_k} \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)}} \mathrm{d}x \leq C(1+k^r), \tag{24}$$

where C is a positive constant depending only on $\alpha, \beta, b, p, r, \|j\|_{p'},\ \|f\|_1,\ \|\nabla v_0\|_p$, and $\|v_0\|_\infty$.

Using the function $T_k(u_n)$ for $k > \{\|g\|_\infty, \|\psi\|_\infty\}$, as a test function for the problem associated with (f_n, ψ, g), we obtain

$$\int_\Omega \frac{a(x,\nabla u_n)\cdot \nabla(T_1(u_n - T_k(u_n)))}{(1+|u_n|)^{\theta(p-1)}} \mathrm{d}x + \int_\Omega b|u_n|^{r-2}u_n T_1(u_n - T_k(u_n))\mathrm{d}x \leq \int_\Omega f_n T_1(u_n - T_k(u_n))\mathrm{d}x,$$

which and (2) give

$$\int_{B_k} \frac{\alpha|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)}} \mathrm{d}x + \int_\Omega b|u_n|^{r-2}u_n T_1(u_n - T_k(u_n))\mathrm{d}x \leq \|f_n\|_1 \leq \|f\|_1 + 1.$$

Note that on the set $\{|u_n| \geq k+1\}$, u_n and $T_1(u_n - T_k(u_n))$ have the same sign. Then

$$\begin{aligned}\int_\Omega |u_n|^{r-2}u_n T_1(u_n - T_k(u_n))\mathrm{d}x =& \int_{D_k} |u_n|^{r-2}u_n T_1(u_n - T_k(u_n))\mathrm{d}x + \int_{B_k} |u_n|^{r-2}u_n T_1(u_n - T_k(u_n))\mathrm{d}x \\ &+ \int_{\{|u_n|\geq k+1\}} |u_n|^{r-2}u_n T_1(u_n - T_k(u_n))\mathrm{d}x \\ \geq& \int_{B_k} |u_n|^{r-2}u_n T_1(u_n - T_k(u_n))\mathrm{d}x.\end{aligned}$$

Thus we have

$$\begin{aligned}\int_{B_k} \frac{\alpha|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)}} \mathrm{d}x + \leq& \|f\|_1 + 1 - \int_{B_k} b|u_n|^{r-2}u_n T_1(u_n - T_k(u_n))\mathrm{d}x \\ \leq& \|f\|_1 + 1 + \int_{B_k} b|u_n|^{r-1}\mathrm{d}x \\ \leq& C\left(1 + \left(\int_{B_k} |u_n|^{q^*}\mathrm{d}x\right)^{\frac{r-1}{q^*}} |B_k|^{1-\frac{r-1}{q^*}}\right),\end{aligned} \tag{25}$$

where q is given in (8) and $q^* = \frac{Nq}{N-q}$.
Let $s = \frac{q\theta(p-1)}{p}$. Note that $q < p$ and $\frac{ps}{p-q} < q^*$. For $\forall k > 0$, we estimate $\int_{B_k} |\nabla u_n|^q \mathrm{d}x$ as follows

$$\begin{aligned}\int_{B_k} |\nabla u_n|^q \mathrm{d}x =& \int_{B_k} \frac{|\nabla u_n|^q}{(1+|u_n|)^s} \cdot (1+|u_n|)^s \mathrm{d}x \\ \leq& \left(\int_{B_k} \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)}} \mathrm{d}x\right)^{\frac{q}{p}} \left(\int_{B_k} (1+|u_n|)^{\frac{ps}{p-q}} \mathrm{d}x\right)^{\frac{p-q}{p}} \\ \leq& C\left(\int_{B_k} \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)}} \mathrm{d}x\right)^{\frac{q}{p}} \left(|B_k|^{\frac{p-q}{p}} + \left(\int_{B_k} |u_n|^{\frac{ps}{p-q}} \mathrm{d}x\right)^{\frac{p-q}{p}}\right)\end{aligned}$$

$$
\begin{aligned}
&\le C\left(\int_{B_k}\frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)}}\mathrm{d}x\right)^{\frac{q}{p}}\left(|B_k|^{\frac{p-q}{p}}+\left(\int_{B_k}|u_n|^{q^*}\mathrm{d}x\right)^{\frac{s}{q^*}}|B_k|^{\frac{p-q}{p}-\frac{s}{q^*}}\right)\\
&\le C\left(|B_k|^{\frac{p-q}{p}}+|B_k|^{\frac{p-q}{p}-\frac{s}{q^*}}\left(\int_{B_k}|u_n|^{q^*}\mathrm{d}x\right)^{\frac{s}{q^*}}+|B_k|^{1-p_1}\left(\int_{B_k}|u_n|^{q^*}\mathrm{d}x\right)^{p_1}\right.\\
&\quad\left.+|B_k|^{1-p_2}\left(\int_{B_k}|u_n|^{q^*}\mathrm{d}x\right)^{p_2}\right)\quad \text{by (25)}\\
&=C\left(|B_k|^{\frac{p-q}{p}}+|B_k|^{\frac{p-q}{p}-\frac{s}{q^*}}\left(\int_{B_k}|u_n|^{q^*}\mathrm{d}x\right)^{\frac{s}{q^*}}\right.\\
&\quad+|B_k|^{1-p_1-C_1}|B_k|^{C_1}\left(\int_{B_k}|u_n|^{q^*}\mathrm{d}x\right)^{p_1}\\
&\quad\left.+|B_k|^{1-p_2-C_2}|B_k|^{C_2}\left(\int_{B_k}|u_n|^{q^*}\mathrm{d}x\right)^{p_2}\right),
\end{aligned}
$$

where $p_1=\frac{q}{p}\frac{r-1}{q^*}, p_2=\frac{s}{q^*}+\frac{q}{p}\frac{r-1}{q^*}$, C_1 and C_2 are positive constants to be chosen later. Note that $\theta<\frac{p-r}{p-1}$, it follows

$$
\frac{\theta(p-1)}{p}+\frac{r-1}{p}<\frac{p-1}{p}<1-\frac{1}{N}=1-\frac{1}{q}+\frac{1}{q^*}.
$$

Thus

$$
\frac{\theta q(p-1)}{p}+\frac{q(r-1)}{p}+1<q+\frac{q}{q^*}\Leftrightarrow s+\frac{q(r-1)}{p}+1<q+\frac{q}{q^*}\Leftrightarrow p_2+\frac{1-p_2}{q^*+1}<\frac{q}{q^*}.
$$

Note that $p_1<p_2<1$. Then for $i=1,2$, we always have

$$
p_i+\frac{1-p_i}{q^*+1}<\frac{q}{q^*}<1.
$$

From this, we may find positive $C_i(i=1,2)$ such that

$$
p_i+\frac{1-p_i}{q^*+1}<p_i+C_i<\frac{q}{q^*}<1,\quad i=1,2. \tag{26}
$$

It follows

$$
\frac{1-p_i}{q^*+1}<C_i\Leftrightarrow 1-p_i-C_i<C_iq^*,\quad i=1,2,
$$

which implies

$$
C_i\alpha_iq^*=\frac{C_iq^*}{1-p_i-C_i}>1,\quad i=1,2, \tag{27}
$$

with $\alpha_i=\frac{1}{1-p_i-C_i}>1,\ i=1,2$. Let $\beta_i=\frac{1}{p_i+C_i}>1, i=1,2$. Then we have $\frac{1}{\alpha_i}+\frac{1}{\beta_i}=1(i=1,2)$. Since $|B_k|\le\frac{1}{k^{q^*}}\int_{B_k}|u_n|^{q^*}\mathrm{d}x$, $|B_k|^{1-p_1-C_1}\le|\Omega|^{1-p_1-C_1}$ and $|B_k|^{1-p_2-C_2}\le|\Omega|^{1-p_2-C_2}$, we have for $k\ge k_0\ge 1$

$$
\int_{B_k}|\nabla u_n|^q\mathrm{d}x\le\frac{C}{k^{q^*\left(\frac{p-q}{p}-\frac{s}{q^*}\right)}}\left(\int_{B_k}|u_n|^{q^*}\mathrm{d}x\right)^{\frac{p-q}{p}}+\frac{C}{k^{q^*C_1}}\left(\int_{B_k}|u_n|^{q^*}\mathrm{d}x\right)^{p_1+C_1}+\frac{C}{k^{q^*C_2}}\left(\int_{B_k}|u_n|^{q^*}\mathrm{d}x\right)^{p_2+C_2}.
$$

Summing up from $k = k_0$ to $k = K$ and using Hölder's inequality, one has

$$\begin{aligned}
\sum_{k=k_0}^{K}\int_{B_k}|\nabla u_n|^q\mathrm{d}x \le& C\Big(\sum_{k=k_0}^{K}\frac{1}{k^{q^*(\frac{p-q}{p}-\frac{s}{q^*})\frac{p}{q}}}\Big)^{\frac{q}{p}}\cdot\Big(\sum_{k=k_0}^{K}\int_{B_k}|u_n|^{q^*}\mathrm{d}x\Big)^{\frac{p-q}{p}}\\
&+C\Big(\sum_{k=k_0}^{K}\frac{1}{k^{q^*C_1\alpha_1}}\Big)^{\frac{1}{\alpha_1}}\cdot\Big(\sum_{k=k_0}^{K}\Big(\int_{B_k}|u_n|^{q^*}\mathrm{d}x\Big)^{\beta_1(p_1+C_1)}\Big)^{\frac{1}{\beta_1}}\\
&+C\Big(\sum_{k=k_0}^{K}\frac{1}{k^{q^*C_2\alpha_2}}\Big)^{\frac{1}{\alpha_2}}\cdot\Big(\sum_{k=k_0}^{K}\Big(\int_{B_k}|u_n|^{q^*}\mathrm{d}x\Big)^{\beta_2(p_2+C_2)}\Big)^{\frac{1}{\beta_2}}\\
=&C\Big(\sum_{k=k_0}^{K}\frac{1}{k^{q^*(\frac{p-q}{p}-\frac{s}{q^*})\frac{p}{q}}}\Big)^{\frac{q}{p}}\cdot\Big(\sum_{k=k_0}^{K}\int_{B_k}|u_n|^{q^*}\mathrm{d}x\Big)^{\frac{p-q}{p}}\\
&+C\Big(\sum_{k=k_0}^{K}\frac{1}{k^{q^*C_1\alpha_1}}\Big)^{\frac{1}{\alpha_1}}\cdot\Big(\sum_{k=k_0}^{K}\int_{B_k}|u_n|^{q^*}\mathrm{d}x\Big)^{p_1+C_1}\\
&+C\Big(\sum_{k=k_0}^{K}\frac{1}{k^{q^*C_2\alpha_2}}\Big)^{\frac{1}{\alpha_2}}\cdot\Big(\sum_{k=k_0}^{K}\int_{B_k}|u_n|^{q^*}\mathrm{d}x\Big)^{p_2+C_2}.
\end{aligned}\tag{28}$$

Note that

$$\int_{\{|u_n|\le K\}}|\nabla u_n|^q\mathrm{d}x=\int_{D_{k_0}}|\nabla u_n|^q\mathrm{d}x+\sum_{k=k_0}^{K}\int_{B_k}|\nabla u_n|^q\mathrm{d}x.\tag{29}$$

To estimate the first integral in the right-hand side of (29), we compute by using Hölder's inequality and (24), obtaining

$$\begin{aligned}
\int_{D_{k_0}}|\nabla u_n|^q\mathrm{d}x \le&\Big(\int_{D_{k_0}}\frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)}}\mathrm{d}x\Big)^{\frac{q}{p}}\Big(\int_{D_{k_0}}(1+|u_n|)^{\frac{ps}{p-q}}\mathrm{d}x\Big)^{\frac{p-q}{p}}\\
\le& C,
\end{aligned}\tag{30}$$

where C depending only on $\alpha,\beta,b,p,\theta,\|j\|_{p'},\|f\|_1,\|\nabla v_0\|_p,\ \|v_0\|_\infty$ and k_0.

Note that $\sum\limits_{k=k_0}^{K}\frac{1}{k^{q^*(\frac{p-q}{p}-\frac{s}{q^*})\frac{p}{q}}}$ and $\sum\limits_{k=k_0}^{K}\frac{1}{k^{q^*C_i\alpha_i}}$ converge due to the fact that $q^*(\frac{p-q}{p}-\frac{s}{q^*})\frac{p}{q}>1$ and $q^*C_i\alpha_i>1$ by (27), respectively. Combining (28)-(30), we get for k_0 large enough

$$\begin{aligned}
\int_{\{|u_n|\le K\}}|\nabla u_n|^q\mathrm{d}x \le& C+C\Big(\int_{\{|u_n|\le K\}}|u_n|^{q^*}\mathrm{d}x\Big)^{\frac{p-q}{p}}+C\Big(\int_{\{|u_n|\le K\}}|u_n|^{q^*}\mathrm{d}x\Big)^{p_1+C_1}\\
&+C\Big(\int_{\{|u_n|\le K\}}|u_n|^{q^*}\mathrm{d}x\Big)^{p_2+C_2}.
\end{aligned}\tag{31}$$

Since $p>q$, $T_K(u_n)\in W^{1,q}(\Omega)$, $T_K(g)=g\in W^{1,q}(\Omega)$ for $K>\|g\|_\infty$. Hence $T_K(u_n)-g\in W_0^{1,q}(\Omega)$. Using the Sobolev embedding $W_0^{1,q}(\Omega)\subset L^{q^*}(\Omega)$ and Poincaré inequality, we obtain

$$\begin{aligned}
\|T_K(u_n)\|_{q^*}^q \le& 2^{q-1}(\|T_K(u_n)-g\|_{q^*}^q+\|g\|_{q^*}^q)\\
\le& C(\|\nabla(T_K(u_n)-g)\|_q^q+\|g\|_{q^*}^q)\\
\le& C(\|\nabla T_K(u_n)\|_q^q+\|\nabla g\|_q^q+\|g\|_{q^*}^q)\\
\le& C\Big(1+\int_{\{|u_n|\le K\}}|\nabla u_n|^q\mathrm{d}x\Big).
\end{aligned}\tag{32}$$

Using the fact that

$$\int_{\{|u_n|\le K\}} |u_n|^{q^*}\mathrm{d}x \le \int_{\{|u_n|\le K\}} |T_K(u_n)|^{q^*}\mathrm{d}x \le \|T_K(u_n)\|_{q^*}^{q^*}, \tag{33}$$

we obtain from (31)-(32),

$$\begin{aligned}\int_{\{|u_n|\le K\}} |\nabla u_n|^q\mathrm{d}x \le& C + C\Big(1+\int_{\{|u_n|\le K\}}|\nabla u_n|^q\mathrm{d}x\Big)^{\frac{q^*}{q}\frac{p-q}{p}} + C\Big(1+\int_{\{|u_n|\le K\}}|\nabla u_n|^q\mathrm{d}x\Big)^{(p_1+C_1)\frac{q^*}{q}}\\ &+ C\Big(1+\int_{\{|u_n|\le K\}}|\nabla u_n|^q\mathrm{d}x\Big)^{(p_2+C_2)\frac{q^*}{q}}.\end{aligned} \tag{34}$$

Note that $p<N \Leftrightarrow \frac{q^*}{q}\frac{p-q}{p}<1$ and $(p_i+C_i)\frac{q^*}{q}<1$ by (26). It follows from (34) that for k_0 large enough, $\int_{\{|u_n|\le K\}}|\nabla u_n|^q\mathrm{d}x$ is bounded independently of n and K. Using (32) and (33), we deduce that $\int_{\{|u_n|\le K\}}|u_n|^{q^*}\mathrm{d}x$ is also bounded independently of n and K. Letting $K\to\infty$, we deduce that $\|\nabla u_n\|_q$ and $\|u_n\|_{q^*}$ are uniformly bounded independently of n. Particularly, u_n is bounded in $W^{1,q}(\Omega)$. Therefore, there exists a subsequence of $\{u_n\}$ and a function $v\in W^{1,q}(\Omega)$ such that $u_n \rightharpoonup v$ weakly in $W^{1,q}(\Omega)$, $u_n\to v$ strongly in $L^q(\Omega)$ and a.e. in Ω. By Lemma 3, $u_n\to u$ in measure in Ω, we conclude that $u=v$ and $u\in W^{1,q}(\Omega)$. ■

Lemma 6 *There exists a subsequence of $\{u_n\}$ and a measurable function u such that ∇u_n converges almost everywhere in Ω to ∇u.*

Proof The proof is quite similar to Theorem 4.1 in [1], which can be also found in [5]. Here we sketch only the main steps due to slight modifications. For $r_2>1$, let $\lambda=\frac{q}{pr_2}<1$, where q is the same as in Lemma 5. Define $A(x,u,\xi)=\frac{a(x,\xi)}{(1+|u|)^{\theta(p-1)}}$ (for the sake of simplicity, we omit the dependence of $A(x,u,\xi)$ on x) and

$$I(n)=\int_\Omega((A(u_n,\nabla u_n)-A(u_n,\nabla u))\cdot\nabla(u_n-u))^\lambda\mathrm{d}x.$$

We fix $k>0$ and split the integral in $I(n)$ on the sets $\{|u|>k\}$ and $\{|u|\le k\}$, obtaining

$$I_1(n,k)=\int_{\{|u|>k\}}((A(u_n,\nabla u_n)-A(u_n,\nabla u))\cdot\nabla(u_n-u))^\lambda\mathrm{d}x,$$

and

$$I_2(n,k)=\int_{\{|u|\le k\}}((A(u_n,\nabla u_n)-A(u_n,\nabla u))\cdot\nabla(u_n-u))^\lambda\mathrm{d}x.$$

For $I_2(n,k)$, one has

$$I_2(n,k)\le I_3(n,k)=\int_\Omega((A_n(u_n,\nabla u_n)-A_n(u_n,\nabla T_k(u)))\cdot\nabla(u_n-T_k(u)))^\lambda dx.$$

Fix $h>0$ and split $I_3(n,k)$ on the sets $\{|u_n-T_k(u)|>h\}$ and $\{|u_n-T_k(u)|\le h\}$, obtaining

$$I_4(n,k,h)=\int_{\{|u_n-T_k(u)|>h\}}((A_n(u_n,\nabla u_n)-A_n(u_n,\nabla T_k(u)))\cdot\nabla(u_n-T_k(u)))^\lambda\mathrm{d}x,$$

and

$$I_5(n,k,h)=\int_{\{|u_n-T_k(u)|\le h\}}((A_n(u_n,\nabla u_n)-A_n(u_n,\nabla T_k(u)))\cdot\nabla(u_n-T_k(u)))^\lambda\mathrm{d}x$$

$$
\begin{aligned}
&= \int_\Omega ((A_n(u_n, \nabla u_n) - A_n(u_n, \nabla T_k(u))) \cdot \nabla T_h(u_n - T_k(u)))^\lambda \mathrm{d}x \\
&\leq |\Omega|^{1-\lambda} \bigg(\int_\Omega (A_n(u_n, \nabla u_n) - A_n(u_n, \nabla T_k(u))) \cdot \nabla T_h(u_n - T_k(u)) \mathrm{d}x \bigg)^\lambda \\
&= |\Omega|^{1-\lambda} \big(I_6(n,k,h) \big)^\lambda .
\end{aligned}
$$

For $I_6(n,k,h)$, it can be split as the difference $I_7(n,k,h) - I_8(n,k,h)$ where

$$I_7(n,k,h) = \int_\Omega A(u_n, \nabla u_n) \cdot \nabla T_h(u_n - T_k(u)) \mathrm{d}x,$$

and

$$I_8(n,k,h) = \int_\Omega A(u_n, \nabla T_k(u)) \cdot \nabla T_h(u_n - T_k(u)) \mathrm{d}x.$$

Note that $|\nabla u_n|$ is bounded in $L^q(\Omega)$ and $\lambda p r_2 = q$. Thanks to Lemma 3 and Lemma 5, one may get in the same way as Theorem 4.1 in [1] that

$$\lim_{k\to\infty} \limsup_{n\to\infty} I_1(n,k) = 0, \lim_{h\to\infty} \limsup_{k\to\infty} \limsup_{n\to\infty} I_4(n,k,h) = 0, \lim_{n\to\infty} I_8(n,k,h) = 0.$$

For $I_7(n,k,h)$, let $k > \max\{\|g\|_\infty, \|\psi\|_\infty\}$ and $n \geq h+k$. Take $T_k(u)$ as a test function for (7), obtaining

$$I_7(n,k,h) \leq \int_\Omega f_n T_h(u_n - T_k(u)) \mathrm{d}x + \int_\Omega b|u_n|^{r-2} u_n T_h(u_n - T_k(u)) \mathrm{d}x.$$

Note that $r - 1 < q^*$, and $\int_\Omega |u_n|^{q^*} \mathrm{d}x$ is uniformly bounded (see the proof of Lemma 5), thus $|u_n|$ converges strongly in $L^1(\Omega)$. Therefore we have

$$\lim_{n\to\infty} \int_\Omega |u_n|^{r-2} u_n T_h(u_n - T_k(u)) \mathrm{d}x = \int_\Omega |u|^{r-2} u T_h(u - T_k(u)) \mathrm{d}x.$$

Then using the strong convergence of f_n in $L^1(\Omega)$, one has

$$\lim_{n\to\infty} I_7(n,k,h) \leq \int_\Omega -f T_h(u - T_k(u)) \mathrm{d}x + \int_\Omega b|u|^{r-2} u T_h(u - T_k(u)) \mathrm{d}x.$$

It follows

$$\lim_{k\to\infty} \lim_{n\to\infty} I_7(n,k,h) \leq 0.$$

Putting together all the limitations and noting that $I(n) \geq 0$, we have

$$\lim_{n\to\infty} I(n) = 0.$$

The same arguments as [1] give that, up to subsequence, $\nabla u_n(x) \to \nabla u(x)$ $a.e.$. ■

Proof of Proposition 4 We shall prove that ∇u_n converges strongly to ∇u_n in $L^q(\Omega)$ for each q, being given by (8). To do that,we will apply Vitalli's Theorem, using the fact that by Lemma 5, ∇u_n is bounded in $L^q(\Omega)$ for each q given by (8). So let $s \in (q, \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)})$ and $E \subset \Omega$ be a measurable set. Then, we have by Hölder's inequality.

$$\int_E |\nabla u_n|^q \mathrm{d}x \leq \bigg(\int_E |\nabla u_n|^r \mathrm{d}x \bigg)^{\frac{q}{s}} \cdot |E|^{\frac{s-q}{s}} \leq C|E|^{\frac{s-q}{s}} \to 0$$

uniformly in n, as $|E| \to 0$. From this and Lemma 6, we deduce that ∇u_n converges strongly to ∇u in $L^q(\Omega)$.

Now assume that $0 \le \theta < \min\{\frac{1}{N-p+1}, \frac{N}{N-1} - \frac{1}{p-1}, \frac{p-r}{p-1}\}$. Note that since ∇u_n converges to ∇u a.e. in Ω, to prove the convergence

$$\frac{a(x,\nabla u_n)}{(1+|u_n|)^{\theta(p-1)}} \to \frac{a(x,\nabla u)}{(1+|u|)^{\theta(p-1)}} \text{ strongly in } (L^1(\Omega))^N,$$

it suffices, thanks to Vitallis Theorem, to show that for every measurable subset $E \subset \Omega$, $\int_E \left|\frac{a(x,\nabla u_n)}{(1+|u_n|)^{\theta(p-1)}}\right| \mathrm{d}x$ converges to 0 uniformly in n, as $|E| \to 0$. Note that $p-1 < \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)})$ by assumptions. For any $q \in (p-1, \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)}))$, we deduce by Hölder's inequality

$$\begin{aligned}\int_E \left|\frac{a(x,\nabla u_n)}{(1+|u_n|)^{\theta(p-1)}}\right| \mathrm{d}x \le& \beta \int_E (j + |\nabla u_n|^{p-1}) \mathrm{d}x \\ \le& \beta \|j\|_{p'} |E|^{\frac{1}{p}} + \beta \left(\int_E |\nabla u_n|^q \mathrm{d}x\right)^{\frac{q}{p-1}} |E|^{\frac{q-p+1}{q}} \\ \to& 0 \text{ uniformly in } n \text{ as } |E| \to 0.\end{aligned}$$

■

Lemma 7 *There exists a subsequence of $\{u_n\}$ such that for all $k > 0$*

$$\frac{a(x,\nabla T_k(u_n))}{(1+|T_k(u_n)|)^{\theta(p-1)}} \to \frac{a(x,\nabla T_k(u))}{(1+|T_k(u)|)^{\theta(p-1)}} \text{ strongly in } (L^1(\Omega))^N.$$

Proof Let k be a positive number. It is well known that if a sequence of measurable functions $\{u_n\}$ with uniformly boundedness in $L^s(\Omega)(s > 1)$ converges in measure to u, then, u_n converges strongly to u in $L^1(\Omega)$. First note that the sequence $\{\frac{a(x,\nabla T_k(u_n))}{(1+|T_k(u_n)|)^{\theta(p-1)}}\}$ is bounded in $L^{p'}(\Omega)$. Indeed, we have by (3) and Lemma 2,

$$\begin{aligned}\int_\Omega \left|\frac{a(x,\nabla T_k(u_n))}{(1+|T_k(u_n)|)^{\theta(p-1)}}\right|^{p'} \mathrm{d}x \le& \beta \|j\|_{p'}^{p'} + \beta \int_\Omega \frac{|\nabla T_k(u_n)|^p}{(1+|T_k(u_n)|)^{\theta p}} \mathrm{d}x \\ \le& \beta \|j\|_{p'}^{p'} + \beta \int_\Omega \frac{|\nabla T_k(u_n)|^p}{(1+|T_k(u_n)|)^{\theta(p-1)}} \mathrm{d}x \\ \le& C.\end{aligned}$$

Next, it suffices to show that there exists a subsequence of $\{u_n\}$ such that

$$\frac{a(x,\nabla T_k(u_n))}{(1+|T_k(u_n)|)^{\theta(p-1)}} \to \frac{a(x,\nabla T_k(u))}{(1+|T_k(u)|)^{\theta(p-1)}} \text{ in measure.}$$

Note that $u_n, u \in W^{1,q}(\Omega)$, where q is the same as in Proposition 4. The a.e. convergence of u_n to u and the fact that $\nabla u_n \to \nabla u$ in measure imply that

$$\nabla T_k(u_n) \to \nabla T_k(u) \text{ in measure.}$$

Let $s,\ t$ be positive numbers and write $\nabla_A u = \frac{a(x,\nabla u)}{(1+|u|)^{\theta(p-1)}}$. Define

$$\begin{aligned}&E_n = \{|\nabla_A T_k(u_n) - \nabla_A T_k(u)| > s\}, \\ &E_n^1 = \{|\nabla T_k(u_n)| > t\}, \\ &E_n^2 = \{|\nabla T_k(u)| > t\},\end{aligned}$$

$$E_n^3 = E_n \cap \{|\nabla T_k(u_n)| \le t\} \cap \{|\nabla T_k(u)| \le t\}.$$

Note that $E_n \subset E_n^1 \cup E_n^2 \cup E_n^3$ for each $n \ge 1$. Using the fact by Lemma 5, the sequence $\{u_n\}$ and the fucntion u are uniformly bounded in $W^{1,q}(\Omega)$, we obtain

$$\mathcal{L}^N(E_n^1) \le \frac{1}{t^q}\int_\Omega |\nabla T_k(u_n)|^q \mathrm{d}x \le \frac{1}{t^q}\int_\Omega |\nabla u_n|^q \mathrm{d}x \le \frac{C}{t^q},$$
$$\mathcal{L}^N(E_n^2) \le \frac{1}{t^q}\int_\Omega |\nabla T_k(u)|^q \mathrm{d}x \le \frac{1}{t^q}\int_\Omega |\nabla u|^q \mathrm{d}x \le \frac{C}{t^q}.$$

We deduce that for any $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$\mathcal{L}^N(E_n^1) + \mathcal{L}^N(E_n^2) < \frac{\varepsilon}{3}, \quad \forall t \ge t_\varepsilon,\ \forall n \ge 1. \tag{35}$$

Note that for $a \ge b \ge 0, \gamma \ge 0$, we have the following inequality

$$\left|\frac{1}{(1+a)^\tau} - \frac{1}{(1+b)^\tau}\right| = \left|\frac{\tau(b-a)}{(1+c)^{1+\tau}}\right| \le \tau|b-a| \quad \text{for some } c \in [b,a].$$

We deduce from (5) and (3) that in E_n^3

$$\begin{aligned}
s <&|\nabla_A T_k(u_n) - \nabla_A T_k(u)| \\
=&\left|\frac{a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u))}{(1+|T_k(u_n)|)^{\theta(p-1)}} + \left(\frac{1}{(1+|T_k(u_n)|)^{\theta(p-1)}} - \frac{1}{(1+|T_k(u)|)^{\theta(p-1)}}\right)a(x,\nabla T_k(u))\right| \\
\le&\theta(p-1)|T_k(u_n) - T_k(u)|\cdot\beta(j + |\nabla T_k(u)|^{p-1}) \\
&+\gamma\begin{cases} |\nabla T_k(u_n) - \nabla T_k(u)|^{p-1}, & \text{if } 1<p<2 \\ |\nabla T_k(u_n) - \nabla T_k(u)|(1+|\nabla T_k(u_n)| + |\nabla T_k(u)|)^{p-2}, & \text{if } p \ge 2 \end{cases} \\
\le&C_0 j|T_k(u_n) - T_k(u)| + C_0(1+t^{p-1}+t^{p-2})(|T_k(u_n) - T_k(u)| + |\nabla T_k(u_n) - \nabla T_k(u)|),
\end{aligned}$$

which leads to $E_n^3 \subset F_1 \cup F_2$, with

$$F_1 = \{j|T_k(u_n) - T_k(u)| > \frac{s}{2C_0}\},$$
$$F_2 = \left\{|T_k(u_n) - T_k(u)| + |\nabla T_k(u_n) - \nabla T_k(u)| > \frac{s}{2C_0(1+t^{p-1}+t^{p-2})}\right\}.$$

In F_1, we have

$$\mathcal{L}^N(F_1) = \frac{2C_0}{s}\int_{F_1}\frac{s}{2C_0}\mathrm{d}x < \frac{2C_0}{s}\int_{F_1} j|T_k(u_n) - T_k(u)|\mathrm{d}x.$$

By Lemma 3, we deduce that there exists $n_0 = n_0(S, C_0, \varepsilon)$ such that

$$\mathcal{L}^N(F_1) \le \frac{\varepsilon}{3},\ \ \forall\, n \ge n_0. \tag{36}$$

For F_2, note that $F^2 \subset F_3 \cup F_4$, with

$$F_3 = \left\{|T_k(u_n) - T_k(u)| > \frac{s}{4C_0(1+t^{p-1}+t^{p-2})}\right\},$$
$$F_4 = \left\{|\nabla T_k(u_n) - \nabla T_k(u)| > \frac{s}{4C_0(1+t^{p-1}+t^{p-2})}\right\}.$$

Using the convergence in measure of $\nabla T_k(u_n)$ to $\nabla T_k(u)$ and $T_k(u_n)$ to $T_k(u)$, for $t = t_\varepsilon$, we obtain the existence of $n_1 = n_1(s, \varepsilon) \geq 1$ such that

$$\mathcal{L}^N(F_2) \leq \mathcal{L}^N(F_3) + \mathcal{L}^N(F_4) < \frac{\varepsilon}{3}, \ \ \forall\, n \geq n_1. \tag{37}$$

Combining (35), (36) and (37), we obtain

$$\mathcal{L}^N(\{|\nabla_A T_k(u_n) - \nabla_A T_k(u)| > s\}) < \varepsilon, \ \ \forall\, n \geq \max\{n_0, n_1\}.$$

Hence the sequence $\{\nabla_A T_k(u_n)\}$ converges in measure to $\nabla_A T_k(u)$ and the lemma follows. ■

3 Proof of the main result

Now we have gathered all the lemmas needed to prove the existence of an entropy solution to the obstacle problem associated with (f, ψ, g). In this part, let f_n be a sequence of smooth functions converging strongly to f in $L^1(\Omega)$, with $\|f_n\|_1 \leq \|f\|_1 + 1$. We consider the sequence of approximated obstacle problems associated with (f_n, ψ, g).

Proof of Theorem 1 Let $v \in K_{g,\psi} \cap L^\infty(\Omega)$. Taking v as a test function in (7) associated with (f_n, ψ, g), we get

$$\int_\Omega \frac{a(x, \nabla u_n)}{(1 + |u_n|)^{\theta(p-1)}} \cdot \nabla(T_t(u_n - v))\mathrm{d}x + \int_\Omega b|u_n|^{r-2}u_n T_t(u_n - v)\mathrm{d}x \leq \int_\Omega f_n T_t(u_n - v)\mathrm{d}x.$$

Since $\{|u_n - v| < t\} \subset \{|u_n| < s\}$ with $s = t + \|v\|_\infty$, the previous inequality can be written as

$$\int_\Omega \chi_n \nabla_A T_s(u_n) \cdot \nabla v\mathrm{d}x \geq \int_\Omega -f_n T_t(u_n - v)\mathrm{d}x + \int_\Omega b|u_n|^{r-2}u_n T_t(u_n - v)\mathrm{d}x + \int_\Omega \chi_n \nabla_A T_s(u_n) \cdot \nabla T_s(u_n)\mathrm{d}x, \tag{38}$$

where $\chi_n = \chi_{\{|u_n - v| < t\}}$ and $\nabla_A u = \frac{a(x,\nabla u)}{(1+|u|)^{\theta(p-1)}}$. It is clear that $\chi_n \rightharpoonup \chi$ weakly* in $L^\infty(\Omega)$. Moreover, χ_n converges a.e. to $\chi_{\{|u-v|<t\}}$ in $\Omega \setminus \{|u - v| = t\}$. It follows that

$$\chi = \begin{cases} 1, \ \text{in } \{|u - v| < t\}, \\ 0, \ \text{in } \{|u - v| > t\}. \end{cases}$$

Note that we have $\mathcal{L}^N(\{|u - v| = t\}) = 0$ for a.e. $t \in (0, \infty)$. So there exists a measurable set $\mathcal{O} \subset (0, \infty)$ such that $\mathcal{L}^N(\{|u - v| = t\}) = 0$ for all $t \in (0, \infty) \setminus \mathcal{O}$. Assume that $t \in (0, \infty) \setminus \mathcal{O}$. Then χ_n converges weakly* in $L^\infty(\Omega)$ and a.e. in Ω to $\chi = \chi_{\{|u-v|<t\}}$. Since $\nabla T_s(u_n)$ converges a.e. to $\nabla T_s(u)$ in Ω (Proposition 4), we obtain by Fatou's Lemma

$$\liminf_{n\to\infty} \int_\Omega \chi_n \nabla_A T_s(u_n) \cdot \nabla T_s(u_n)\mathrm{d}x \geq \int_\Omega \chi \nabla_A T_s(u) \cdot \nabla T_s(u)\mathrm{d}x. \tag{39}$$

Using the strong convergence of $\nabla_A T_s(u_n)$ to $\nabla_A T_s(u)$ in $L^1(\Omega)$ (Lemma 7) and the weak* convergence of χ_n to χ in $L^\infty(\Omega)$, we obtain

$$\lim_{n\to\infty} \int_\Omega \chi_n \nabla_A T_s(u_n) \cdot \nabla v\mathrm{d}x = \int_\Omega \chi \nabla_A T_s(u) \cdot \nabla v\mathrm{d}x. \tag{40}$$

Moreover, due to the strong convergence of f_n to f and $|u_n|^{r-2}u_n$ to $|u|^{r-2}u$ (by $r - 1 < q^*$ and the boundedness of $\|u_n\|_{q^*}$) in $L^1(\Omega)$, and the weak* convergence of $T_t(u_n - v)$ to $T_t(u - v)$ in $L^\infty(\Omega)$, by passing to the limit in (38) and taking into account (39)-(40), we obtain

$$\int_\Omega \chi \nabla_A T_s(u) \cdot \nabla v\mathrm{d}x - \int_\Omega \chi \nabla_A T_s(u) \cdot \nabla T_s(u)\mathrm{d}x \geq \int_\Omega -f T_t(u - v)\mathrm{d}x + \int_\Omega b|u|^{r-2}u T_t(u - v)\mathrm{d}x,$$

which can be written as

$$\int_{\{|v-u|\le t\}} \chi \nabla_A T_s(u)\cdot(\nabla v-\nabla u)\mathrm{d}x \ge \int_\Omega -fT_t(u-v)\mathrm{d}x + \int_\Omega b|u|^{r-2}uT_t(u-v)\mathrm{d}x,$$

or since $\chi=\chi_{\{|u-v|<t\}}$ and $\nabla(T_t(u-v))=\chi_{\{|u-v|<t\}}\nabla(u-v)$

$$\int_\Omega \nabla_A u\cdot\nabla T_t(u-v)\mathrm{d}x + \int_\Omega b|u|^{r-2}uT_t(u-v)\mathrm{d}x \le \int_\Omega fT_t(u-v)\mathrm{d}x, \forall\, t\in(0,\infty)\setminus\mathcal{O}.$$

For $t\in\mathcal{O}$, we know that there exists a sequence $\{t_k\}$ of numbers in $(0,\infty)\setminus\mathcal{O}$ such that $t_k\to t$ due to $|\mathcal{O}|=0$. Therefore, we have

$$\int_\Omega \nabla_A u\cdot\nabla T_{t_k}(u-v)\mathrm{d}x + \int_\Omega b|u|^{r-2}uT_{t_k}(u-v)\mathrm{d}x \le \int_\Omega fT_{t_k}(u-v)\mathrm{d}x. \tag{41}$$

Since $\nabla(u-v)=0$ a.e. in $\{|u-v|=t\}$, the left-hand side of (41) can be written as

$$\int_\Omega \nabla_A u\cdot\nabla T_{t_k}(u-v)\mathrm{d}x = \int_{\Omega\setminus\{|u-v|=t\}} \chi_{\{|u-v|<t_k\}}\nabla_A u\cdot\nabla(u-v)\mathrm{d}x.$$

The sequence $\chi_{\{|u-v|<t_k\}}$ converges to $\chi_{\{|u-v|<t\}}$ a.e. in $\Omega\setminus\{|u-v|=t\}$ and therefore converges weakly* in $L^\infty(\Omega\setminus\{|u-v|=t\})$. We obtain

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega \nabla_A u\cdot\nabla T_{t_k}(u-v)\mathrm{d}x &= \int_{\Omega\setminus\{|u-v|=t\}} \chi_{\{|u-v|<t\}}\nabla_A u\cdot\nabla(u-v)\mathrm{d}x\\ &= \int_\Omega \chi_{\{|u-v|<t\}}\nabla_A u\cdot\nabla(u-v)\mathrm{d}x\\ &= \int_\Omega \nabla_A u\cdot\nabla T_t(u-v)\mathrm{d}x.\end{aligned} \tag{42}$$

For the right-hand side of (41), we have

$$\left|\int_\Omega fT_{t_k}(u-v)\mathrm{d}x - \int_\Omega fT_t(u-v)\mathrm{d}x\right| \le |t_k-t|\cdot\|f\|_1 \to 0 \ \text{ as } k\to\infty. \tag{43}$$

Similarly, we have

$$\left|\int_\Omega |u|^{r-2}uT_{t_k}(u-v)\mathrm{d}x - \int_\Omega |u|^{r-2}uT_t(u-v)\mathrm{d}x\right| \le |t_k-t|\cdot\||u|^{r-1}\|_1 \to 0 \text{ as } k\to\infty. \tag{44}$$

It follows from (41)-(44) that we have the inequality

$$\int_\Omega \nabla_A u\cdot\nabla T_t(u-v)\mathrm{d}x + \int_\Omega b|u|^{r-2}uT_t(u-v)\mathrm{d}x \le \int_\Omega fT_t(u-v)\mathrm{d}x,\ \forall\, t\in(0,\infty).$$

Hence, u is an entropy solution of the obstacle problem associated with (f,ψ,g). The dependence of the entropy solution on the data $f\in L^1(\Omega)$ is guaranteed by Proposition 4. ■

4 Declarations

4.1 Availability of data and material

Not applicable.

4.2 *Competing interests*

The author declares that he has no competing interests.

4.3 *Funding*

This research was partially supported by the Fundamental Research Funds for the Central Universities (No.: 10801X10096022).

4.4 *Authors' contributions*

This paper was completed by JZ independently.

4.5 *Acknowledgements*

Not applicable.

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